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LETTER TO THE EDITOR

An approach for constructing non-isospectral hierarchies of evolution equations

Wen-xiu Ma

CCAST (World Laboratory) PO Box 8730, Beijing, 100080, People's Republic of China and (mailing address) Institute of Mathematics, Fudan University, Shanghai 200433, People's Republic of China

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Abstract. For a given isospectral ($\lambda_t = 0$) hierarchy of evolution equations, we propose a simple method of constructing its corresponding non-isospectral ($\lambda_t = \lambda^n$, $n \geqslant 0$) hierarchy of evolution equations closely related to τ-symmetries. It is crucial to find an initial Lax operator W_0 and an initial vector field g_0 satisfying the key equation $[W_0, L] = L'[g_0] - I$, in which L, I are spectral and identity operators, respectively. As examples, we present the corresponding non-isospectral hierarchies of equations and display the fundamental relations generating symmetry algebras for KdV hierarchy, AKNS hierarchy and a new integrable hierarchy.

It is well known that, starting from a proper linear spectral problem $L\psi = \lambda\psi$ (λ spectral parameter), we can generate a hierarchy of isospectral ($\lambda_t = 0$) evolution equations integrable by the inverse scattering transform (IST) (see, for example, Ablowitz and Segur 1981, Newell 1985, Geng 1990, and Tu 1989a, b). Suppose that the spectral problem is not isospectral, i.e. $\lambda_t \neq 0$, for example, $\lambda_t = \lambda^n (n \geqslant 0)$, we can still generate a hierarchy of corresponding evolution equations (for instance, see Li 1982). Furthermore, evolution equations of this kind are often solved still by IST (see, for example, Calogero and Degasperis 1978).

In this letter, a non-isospectral hierarchy just means a series of evolution equations corresponding to $\lambda_t = \lambda^n$, $n \geqslant 0$. We shall show that after we obtain an isospectral hierarchy, we can generate a non-isospectral hierarchy by a simple and clear approach. In general, the flows of the isospectral and non-isospectral hierarchies constitute a semi-product of a Kac–Moody algebra and the Virasoro algebra. Moreover, we can usually obtain hierarchies of τ-symmetries of the isospectral hierarchy from the non-isospectral hierarchy. We shall present three examples to show those.

For the matrix Schrödinger spectral problem

$$L = -\partial_{xx} + Q(x, t)$$

($Q(x, t)$ is an $N \times N$ matrix) Bruschi and Ragnisco (1980) have constructed a class of non-isospectral evolution equations through some direct computations and extended the Lax method to the particular class of evolution equations. However, our approach is quite different from that of Bruschi and Ragnisco, and is universally applicable, not only to the matrix Schrödinger spectral problem mentioned above but also to any other arbitrary spectral problem. Our example 1 shows a case of $N = 1$ in Bruschi and Ragnisco (1980) and the evolution operators B_n, $n \geqslant 0$, given by our approach, make the series in neater form.

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In the following we give some fundamental symbols and notations. Let $x = (x^1, \ldots, x^p)^T \in R^p$, $t \in R$, $u = (u^1, \ldots, u^q)^T$, $u^i = u^i(x, t)$, $1 \leq i \leq q$. For $\alpha = (\alpha_1, \ldots, \alpha_p)$, $\alpha_i \in Z$, $a_i \geq 0$, $1 \leq i \leq p$, write

$$D^\alpha = \left(\frac{\mathrm{d}}{\mathrm{d}x^1}\right)^{\alpha_1} \cdots \left(\frac{\mathrm{d}}{\mathrm{d}x^p}\right)^{\alpha_p} \qquad |\alpha| = \sum_{i=1}^{p} \alpha_i.$$

We denote by $\mathscr{B}$ all complex (or real) functions $P[u] = P(x, t, u)$ which are C^∞-differentiable with respect to x, t and C^∞-Gateaux differentiable with respect to $u = u(x)$ (as functions of x), and let $\mathscr{B}^r = \{(P_1, \ldots, P_r)^T | P_i \in \mathscr{B}, 1 \leq i \leq r\}$. We denote by $\mathscr{V}^r$ all linear operators $\Phi = \Phi(x, t, u): \mathscr{B}^r \to \mathscr{B}^r$ which are C^∞-differentiable with respect to x, t and C^∞-Gateaux differentiable with respect to $u = u(x)$, and by $\mathscr{V}_0^r$ all matrix differential operators $L = L(x, t, u): \mathscr{B}^r \to \mathscr{B}^r$ with the following form

$$L = (L_{ij})_{r \times r} \qquad L_{ij} = \sum_{|\alpha| \leq \alpha(i,j)} P_\alpha^{ij}[u] D^\alpha \qquad P_\alpha^{ij}[u] \in \mathscr{B}. \tag{1}$$

For two vector fields $X, Y \in \mathscr{B}^q$, define their product $[X, Y] \in \mathscr{B}^q$ as follows

$$[X, Y] = X'[Y] - Y'[X] = \frac{\partial}{\partial \varepsilon}(X(u + \varepsilon Y) - Y(u + \varepsilon X))|_{\varepsilon=0}. \tag{2}$$

Bowman (1987) has shown that $\langle \mathscr{B}^q, [\cdot, \cdot] \rangle$ constitutes a Lie algebra. The Gateaux derivative operator $\Phi': \mathscr{B}^q \to \mathscr{V}^r$ of an operator $\Phi \in \mathscr{V}^r$ is defined by

$$\Phi'[X]Y = \frac{\partial}{\partial \varepsilon} \Phi(u + \varepsilon X) Y|_{\varepsilon=0} \qquad X \in \mathscr{B}^q \quad Y \in \mathscr{B}^r. \tag{3}$$

For $\Phi \in \mathscr{V}^q$, $X \in \mathscr{B}^q$, the Lie derivative $L_X \Phi \in \mathscr{V}^q$ is defined as follows

$$(L_X \Phi) Y = \Phi[X, Y] - [X, \Phi Y] \qquad Y \in \mathscr{B}^q. \tag{4}$$

Furthermore it may be shown that

$$L_X \Phi = \Phi'[X] - [X', \Phi] = \Phi'[X] - X'\Phi + \Phi X'. \tag{5}$$

This kind of Lie derivative has an explicit geometrical meaning (see Magri 1980).

Let a spectral operator $L = L(x, u) \in \mathscr{V}_0^r$ and its Gateaux derivative operator $L': \mathscr{B}^q \to \mathscr{V}_0^r$ be an injective homomorphism. We consider the following linear spectral problem

$$L\psi = \lambda\psi \qquad \lambda \text{ is a spectral parameter.} \tag{6}$$

Suppose that the spectral problem (6) and a series of auxiliary problems

$$\psi_t = A_m \psi \qquad A_m \in \mathscr{V}^r \qquad m \geq 0 \tag{7}$$

determine an isospectral hierarchy of integrable evolution equations

$$u_t = K_m = \Phi^m f_0 \qquad \Phi \in \mathscr{V}^q \qquad f_0 \in \mathscr{B}^q \qquad m \geq 0. \tag{8}$$

We first look for a pair of solutions $W_0 \in \mathscr{V}^r$, $g_0 \in \mathscr{B}^q$ of the following equation

$$[W_0, L] = L'[g_0] - I \tag{9}$$

where I is the identity operator from $\mathscr{B}^r$ to $\mathscr{B}^r$. And then we can work out a hierarchy of evolution equations

$$u_t = \sigma_n = \Phi^n g_0 \qquad n \geq 0 \tag{10}$$

where the operator Φ is defined as in (8). We shall explain that the hierarchy (10) is generally a non-isospectral hierarchy of equations corresponding to $\lambda_t = \lambda^n$, $n \geq 0$.

For any given vector field $X \in \mathscr{B}^q$, we construct an operator equation of $V \in \mathscr{V}^r$

$$[V, L] = L'[\Phi X] - L'[X]L. \tag{11}$$

Choose an operator solution $V = V(X)$ of the equation (11). Set $W_{j+1} = V(\sigma_j), j \geq 0$, and then we have

$$[W_{j+1}, L] = L'[\sigma_{j+1}] - L'[\sigma_j]L \qquad j \geq 0. \tag{12}$$

Further set $B_n = \sum_{j=0}^n W_j L^{n-j}$, $n \geq 0$. By (9) and (12), we can calculate $[B_n, L]$ as follows

$$\begin{aligned}[B_n, L] &= \left[\sum_{j=0}^{n} W_j L^{n-j}, L\right] \\ &= \sum_{j=0}^{n} [W_j, L] L^{n-j} \\ &= (L'[\sigma_0] - I)L^n + \sum_{j=1}^{n} (L'[\sigma_j] - L'[\sigma_{j-1}])L^{n-j} \\ &= L'[\sigma_n] - L^n \qquad n \geq 0.\end{aligned}$$

In this way, for any $n \geq 0$, the evolution equation $u_t = \sigma_n$ is, by $L_t = L'[u_t]$, the compatibility condition of the following problems

$$\begin{aligned} L\psi &= \lambda\psi \qquad \lambda_t = \lambda^n \\ \psi_t &= B_n \psi. \end{aligned} \tag{13}$$

This shows the hierarchy (10) is just a non-isospectral hierarchy of evolution equations corresponding to $\lambda_t = \lambda^n$, $n \geq 0$.

By now, we have performed some formal manipulations with the non-isospectral hierarchy (10). In the above skeleton, we only desire that there exists any solution of the operator equation (11). Generally, this condition is included in the existence of the isospectral hierarchy (8) and thus does not raise any new requirements.

Besides, we point out that the operator $\Phi \in \mathscr{V}^r$ is usually a hereditary symmetry and often satisfies the following fundamental relations

$$L_{f_0}\Phi = 0 \qquad L_{g_0}\Phi = \beta \qquad \Phi[f_0, g_0] = [f_0, \Phi g_0] = \gamma f_0 \tag{14}$$

where $\beta = \sum \beta_i \Phi^i$, $\gamma = \sum \gamma_i \Phi^i$ are two constant coefficient polynomials of the operator Φ. Based on (14) and according to the result of Tu (1989b), Ma (1990) or Oevel (1987), we can show that $\{\sigma_m = \Phi^m g_0\}_{m=0}^{\infty}$ is a common hierarchy of the first-order master-symmetries (see Fuchssteiner 1983 for definition) for the isospectral hierarchy (8) and can further give the corresponding symmetry algebras. Next we shall not only solve the key equation (9), but also display the solutions of the operator equation (11) and the relations (14) of the triple (Φ, f_0, g_0) for the KdV hierarchy, AKNS hierarchy and a new integrable hierarchy of equations.

Example 1. We first consider the KdV hierarchy of equations

$$u_t = K_m = \Phi^m f_0 = \Phi^m u_x \qquad x, t \in R \qquad m \geq 0 \tag{15}$$

where

$$\Phi = \partial^2 + 4u + 2u_x\partial^{-1} \qquad \partial = \frac{\mathrm{d}}{\mathrm{d}x} \tag{16}$$

which is a hereditary symmetry. The hierarchy (15) corresponds to the following spectral operator

$$L = 4\partial^2 + 4u. \tag{17}$$

Obviously, we have $L'[X] = 4X$, $X \in \mathscr{B}$. Thus L' is injective and the corresponding equation (9) reads as

$$[W_0, L] = L'[g_0] - 1 = 4g_0 - 1. \tag{18}$$

Choosing $W_0 = P + Q\partial$, $P, Q \in \mathscr{B}$, we have

$$[W_0, L] = (-4P_{xx} + 4u_xQ + 8uQ_x) - 4(2P_x + Q_{xx})\partial - 2Q_xL.$$

From this, we can easily obtain a pair of solutions W_0, g_0 of the equation (18)

$$W_0 = d_1 + d_2\partial \qquad g_0 = \tfrac{1}{4} + d_2u_x \tag{19}$$

where d_1, d_2 are arbitrary constants. Moreover it is not difficult to show that the corresponding operator equation (11) has the following special solution

$$V = V(X) = -X + 2(\partial^{-1}X)\partial \qquad X \in \mathscr{B}.$$

Therefore the hierarchy of evolution equations

$$u_t = \sigma_n = \Phi^n g_0 = \Phi^n(\tfrac{1}{4} + d_2u_x) \qquad n \geqslant 0 \tag{20}$$

is a hierarchy of non-isospectral equations with $\lambda_t = \lambda^n$, $n \geqslant 0$. In addition, it is easy to show that

$$L_{f_0}\Phi = 0 \qquad L_{g_0}\Phi = 1 \qquad \Phi[f_0, g_0] = [f_0, \Phi g_0] = [u_x, u + \tfrac{1}{2}xu_x] = \tfrac{1}{2}f_0.$$

Example 2. Next we consider the AKNS hierarchy of equations

$$u_t = \begin{bmatrix} q \\ r \end{bmatrix}_t = K_m = \Phi^m f_0 = \Phi^m \begin{bmatrix} -q \\ r \end{bmatrix} \qquad m \geqslant 0 \tag{21}$$

with

$$\Phi = \begin{bmatrix} -\frac{1}{2}\partial + q\partial^{-1}r & q\partial^{-1}q \\ -r\partial^{-1}r & \frac{1}{2}\partial - r\partial^{-1}q \end{bmatrix} \qquad \partial = \frac{\mathrm{d}}{\mathrm{d}x} \tag{22}$$

which is a hereditary symmetry. The AKNS hierarchy corresponds to the following spectral problem

$$L\psi = \lambda\psi \qquad L = \begin{bmatrix} -\partial & q \\ -r & \partial \end{bmatrix}. \tag{23}$$

Evidently, we have

$$L'[X] = \begin{bmatrix} 0 & X_1 \\ -X_2 & 0 \end{bmatrix} \qquad X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \in \mathscr{B}^2.$$

Thus L' is injective and the corresponding equation (9) becomes

$$[W_0, L] = L'[g_0] - I = \begin{bmatrix} -1 & g_{01} \\ -g_{02} & -1 \end{bmatrix} \qquad g_0 = \begin{bmatrix} g_{01} \\ g_{02} \end{bmatrix} \in \mathscr{B}^2. \tag{24}$$

Choosing $W_0 = [\begin{smallmatrix} P_1 & P_2 \\ P_3 & P_4 \end{smallmatrix}]$, $P_i \in \mathscr{B}$, $1 \leqslant i \leqslant 4$, we have

$$[W_0, L] = \begin{bmatrix} P_{1x} - rP_2 - qP_3 & P_{2x} + q(P_1 - P_4) \\ -P_{3x} + r(P_1 - P_4) & -P_{4x} + rP_2 + qP_3 \end{bmatrix} + \begin{bmatrix} 0 & 2P_2 \\ -2P_3 & 0 \end{bmatrix}\partial.$$

In this way, by (24) we obtain

$$P_2 = P_3 = 0 \qquad P_{1x} = -1 \qquad P_{4x} = 1.$$

and further obtain a pair of solutions of the equation (24)

$$W_0 = \begin{bmatrix} -x+d_1 & 0 \\ 0 & x+d_2 \end{bmatrix} \qquad g_0 = (2x+d)\begin{bmatrix} -q \\ r \end{bmatrix} \tag{25}$$

where d_1, d_2 are arbitrary constants and $d = d_2 - d_1$. Moreover by a direct calculation, we may show that the corresponding operator equation (11) has the following special solution

$$V = V(X) = \frac{1}{2}\begin{bmatrix} \partial^{-1}(rX_1 + qX_2) & -X_1 \\ X_2 & -\partial^{-1}(rX_1 + qX_2) \end{bmatrix} \qquad X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \in \mathscr{B}^2.$$

Therefore the following hierarchy of evolution equations

$$u_t = \sigma_n = \Phi^n g_0 = \Phi^n (2x+d)\begin{bmatrix} -q \\ r \end{bmatrix} \qquad n \geqslant 0 \tag{26}$$

is a hierarchy of non-isospectral equations with $\lambda_t = \lambda^n$, $n \geqslant 0$. In addition, we easily show that

$$L_{f_0}\Phi = 0 \qquad L_{g_0}\Phi = 1$$

$$\Phi[f_0, g_0] = [f_0, \Phi g_0] = [f_0, \Phi(-2xq, 2xr)^{\mathrm{T}}] = [f_0, (q + xq_x, r + xr_x)^{\mathrm{T}}] = 0.$$

Example 3. Finally we consider a new hierarchy of integrable evolution equations in Ma (1992a) ($p = 1$, $q = 2$)

$$u_t = \begin{bmatrix} r \\ s \end{bmatrix}_t = K_m = \Phi^m f_0 = \Phi^m \begin{bmatrix} 2\alpha^{-1} r_x \\ 2\alpha^{-1} s_x \end{bmatrix} \qquad \alpha = \alpha_1 - \alpha_2 \qquad m \geqslant 0 \tag{27}$$

with the hereditary symmetry

$$\Phi = \begin{bmatrix} \alpha^{-1}[\partial - (\alpha_4 - 1)s] & -\alpha^{-1}(\alpha_4 - 1)(\partial r \partial^{-1} + r) \\ \dfrac{2\alpha_3}{\alpha(\alpha_4 - 1)} & -\alpha^{-1}[\partial + (\alpha_4 - 1)\partial s \partial^{-1}] \end{bmatrix} \qquad \partial = \frac{\mathrm{d}}{\mathrm{d}x} \tag{28}$$

in which α_i, $1 \leqslant i \leqslant 4$, are constants and $\alpha_1\alpha_2(\alpha_1 - \alpha_2)\alpha_3(\alpha_4 - 1) \neq 0$. The first nonlinear system in the hierarchy (27) is as follows

$$r_t = -2\alpha^{-2}[-r_{xx} + 2(\alpha_4 - 1)(rs)_x]$$

$$s_t = -2\alpha^{-2}\left[-\frac{2\alpha_3}{\alpha_4 - 1} r_x + s_{xx} + 2(\alpha_1 - 1)ss_x\right].$$

That hierarchy corresponds to the spectral problem

$$\phi_x = \begin{bmatrix} \alpha_1\lambda + \alpha_4 s & r \\ \alpha_3 & \alpha_2\lambda + s \end{bmatrix}\phi$$

which may be rewritten as the following Lax form

$$L\psi = \lambda\psi \qquad L = \begin{bmatrix} \alpha_1^{-1}(\partial - \alpha_4 s) & -\alpha_1^{-1} r \\ -\alpha_2^{-1}\alpha_3 & \alpha_2^{-1}(\partial - s) \end{bmatrix}. \tag{29}$$

Obviously, the Gateaux derivative operator L' reads as

$$L'[X]=\begin{bmatrix}-\alpha_1^{-1}\alpha_4 X_2 & -\alpha_1^{-1}X_1\\ 0 & -\alpha_2^{-1}X_2\end{bmatrix} \qquad X=\begin{bmatrix}X_1\\ X_2\end{bmatrix}\in\mathscr{B}^2$$

and thus L' is injective. In this case, the equation (9) takes the form

$$[W_0, L]=L'[g_0]-I=\begin{bmatrix}-\alpha_1^{-1}\alpha_4 g_{02}-1 & -\alpha_1^{-1}g_{01}\\ 0 & -\alpha_2^{-1}g_{02}-1\end{bmatrix} \qquad g_0=\begin{bmatrix}g_{01}\\ g_{02}\end{bmatrix}\in\mathscr{B}^2. \tag{30}$$

Let $W_0=\left[\begin{smallmatrix}P_1 & P_2\\ P_3 & P_4\end{smallmatrix}\right]$, $P_i\in\mathscr{B}$, $1\leqslant i\leqslant 4$ and we can obtain a pair of solutions of (30):

$$W_0=\frac{\alpha_2\alpha_4-\alpha_1}{\alpha_4-1}\begin{bmatrix}x & 0\\ 0 & x\end{bmatrix} \qquad g_0=\begin{bmatrix}0\\ (\alpha_2-\alpha_1)/(\alpha_4-1)\end{bmatrix}. \tag{31}$$

In addition, it may be shown by a direct calculation that the corresponding operator equation (11) has the first-order differential operator solution with the following special form

$$V=V(X)=\begin{bmatrix}Q_1 & Q_2\\ Q_3 & Q_4\end{bmatrix}+\begin{bmatrix}R_1 & 0\\ 0 & R_2\end{bmatrix}\partial \qquad X=\begin{bmatrix}X_1\\ X_2\end{bmatrix}\in\mathscr{B}^2 \tag{32}$$

with

$$Q_1=-\frac{(\alpha_2\alpha_4-\alpha_1)\alpha_4}{\alpha\alpha_1}s\partial^{-1}X_2-\frac{\alpha_4}{\alpha}X_2+\frac{\alpha_3(\alpha_4+1)}{\alpha(\alpha_1-1)}\partial^{-1}X_1 \tag{33a}$$

$$Q_2=\frac{1}{\alpha}X_1-\frac{(\alpha_2\alpha_4-\alpha_1)}{\alpha\alpha_1}r\partial^{-1}X_2 \qquad Q_3=-\frac{(\alpha_2\alpha_4-\alpha_1)\alpha_3}{\alpha\alpha_2}\partial^{-1}X_2 \tag{33b}$$

$$Q_4=-\frac{(\alpha_2\alpha_4-\alpha_1)}{\alpha\alpha_1}s\partial^{-1}X_2-\frac{1}{\alpha}X_2+\frac{\alpha_3(\alpha_4+1)}{\alpha(\alpha_4-1)}\partial^{-1}X_1 \tag{33c}$$

$$R_1=-\frac{\alpha_4}{\alpha_1}\partial^{-1}X_2 \qquad R_2=-\frac{1}{\alpha_2}\partial^{-1}X_2. \tag{33d}$$

Therefore according to our earlier result, we see that the hierarchy of evolution equations

$$u_t=\sigma_n=\Phi^n g_0=\Phi^n\begin{bmatrix}0\\ \dfrac{\alpha_2-\alpha_1}{\alpha_4-1}\end{bmatrix} \qquad n\geqslant 0 \tag{34}$$

is a hierarchy of non-isospectral equations corresponding to $\lambda_t=\lambda^n$, $n\geqslant 0$.

In this case, it may be similarly proved that

$$L_{f_0}\Phi=0 \qquad L_{g_0}\Phi=1$$

$$\Phi[f_0, g_0]=[f_0, \Phi g_0]=[f_0, (2r+xr_x, s+xs_x)^{\mathrm{T}}]=f_0.$$

Thus by lemma 2 of Ma (1990), we can arrive at

$$[\Phi^m f_0, \Phi^n f_0]=0,\ m, n\geqslant 0$$

$$[\Phi^m f_0, \Phi^n g_0]=(m+1)\Phi^{m+n-1}f_0 \qquad m, n\geqslant 0 \quad m+n\geqslant 1$$

$$[\Phi^m g_0, \Phi^n g_0]=(m-n)\Phi^{m+n-1}g_0 \qquad m, n\geqslant 0.$$

From these, we easily see that the evolution equation $u_t = K_l$ $(l \geqslant 0)$ possesses a hierarchy of K-symmetries $\{K_m\}_{m=0}^{\infty}$ and a hierarchy of τ-symmetries $\{\tau_n^{(l)} = t[K_l, \sigma_n] + \sigma_n\}_{n=0}^{\infty}$, and that these two hierarchies of symmetries constitute an infinite-dimensional Lie algebra (a semi-product of $\mathrm{span}\{K_m \mid m \geqslant 0\}$ and $\mathrm{span}\{\tau_n^{(l)} \mid n \geqslant 0\}$):

$$[K_m, K_n] = 0, \; m, n \geqslant 0$$

$$[K_m, \tau_n^{(l)}] = (m+1)K_{m+n-1} \qquad K_{-1} = 0 \quad m, n \geqslant 0$$

$$[\tau_m^{(l)}, \tau_n^{(l)}] = (m-n)\tau_{m+n-1}^{(l)} \qquad \tau_{-1}^{(l)} = 0 \quad m, n \geqslant 0.$$

Apart from the above results, we have shown in Ma (1992b) that the integrable hierarchy (27) possesses the bi-Hamiltonian structures

$$u_t = K_m = \Phi^m f_0 = J\frac{\delta H_{m+1}}{\delta u} = M\frac{\delta H_m}{\delta u} \qquad m \geqslant 0$$

with the Hamiltonian pair

$$J = \begin{bmatrix} 0 & -\dfrac{\alpha}{\alpha_3(\alpha_4 - 1)}\partial \\ -\dfrac{\alpha}{\alpha_3(\alpha_4 - 1)}\partial & 0 \end{bmatrix}$$

$$M = J\Phi^* = \begin{bmatrix} \alpha_3^{-1}(r\partial + \partial r) & -\dfrac{1}{\alpha_3(\alpha_4 - 1)}\partial^2 + \alpha_3^{-1}s\partial \\ \dfrac{1}{\alpha_3(\alpha_4 - 1)}\partial^2 + \alpha_3^{-1}\partial s & -\dfrac{2}{(\alpha_4 - 1)^2}\partial \end{bmatrix}$$

and the Hamiltonian functions

$$H_m = -\frac{\alpha}{(m+1)}a_{m+2} \qquad m \geqslant 0.$$

Here a_m, $m \geqslant 2$, are determined by the recursion formula

$$\left.\begin{aligned} & b_0 = c_0 = 0 \qquad a_0 = 1 \\ & a_{mx} = rc_m - \alpha_3 b_m \\ & b_{mx} = \alpha b_{m+1} + (\alpha_4 - 1)sb_m - 2ra_m \\ & c_{mx} = -\alpha c_{m+1} - (\alpha_4 - 1)sc_m + 2\alpha_3 a_m \end{aligned}\right\} \quad m \geqslant 0.$$

Moreover there exists a Lax operator algebra, analogous to Lax operator algebras of KdV and AKNS hierarchies (see Cheng and Li 1991, Zhang and Cheng 1990), corresponding to the integrable hierarchy (27), which is left to a later paper.

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